Efficient heat transfer teaching in engineering energy problems

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ABSTRACT

In this paper we discuss the application of the proper eigenvalue method (PEM) developed by the author to time dependent problems in heat transfer not involving the changes of phase of the system components.

The method is based on the solution of the generalized eigenvalue problem, where the eigenvalues are identified as poles and zeros. This process allows a trivial determination of the inverse Laplace transforms involved.

Iteration in time is avoided (thus avoiding possible instabilities in systems with more than one space dimension), achieving a reduction of several orders of magnitude in the required computer time.

The method can also be applied to linear problems in electrochemistry, fluid flow in porous media, electromagnetic theory, and fluid mechanics.

Keywords: fossil fuels, heat transfer, inverse Laplace transforms, modeling, linear PDE

1. INTRODUCTION

As time progresses engineering education is becoming more complex and digital computers are called to play an increasing role as support tools. Of particular importance are those programs that can be run on relatively small computers (or work stations) which are available to most small universities that cannot have access to large scale centralized computing facilities.

This is the case for most universities in Puerto Rico, where the availability of software that can solve fairly complex engineering (or science problems) would be very convenient. For this reason, in the present paper we will describe a very efficient numerical procedure that allows the solution of many linear prtial differential equations. The numerical procedure developed [Callarotti 1995] can determine the transient time response for linear (or linearizable) systems in two or three dimensions avoiding time iterations and saving two or three orders of magnitude of computer time. The program has been successfully applied to the solution of heat transfer problems (with/without convection), electromagnetism, circuits, fluid flow through porous media, electrochemistry and fluid mechanics

Figure 1 indicates some of the possible ways for solving Linear Partial Differential Equations (LPDE). Analytical solutions are available in some cases but even in those simple cases the solutions are obtained in terms of sines, cosines, hyperbolic sines and hyperbolic cosines, Bessel functions, Tchebishev or Legendre polynomials, all of which are evaluated by infinite series.

The second type of solutions requires time iteration, which always works in one dimensional systems, where the stable time increment is known. Two and three dimensional solutions, require guessing the (Δt) that allows a stable iterated solution. Sometimes the solution diverges and the process must be repeated using lower values for (Δt). In one dimension the stable (Δt) is given by:

$$\Delta t < \frac{1}{2(\Delta x)^2}$$

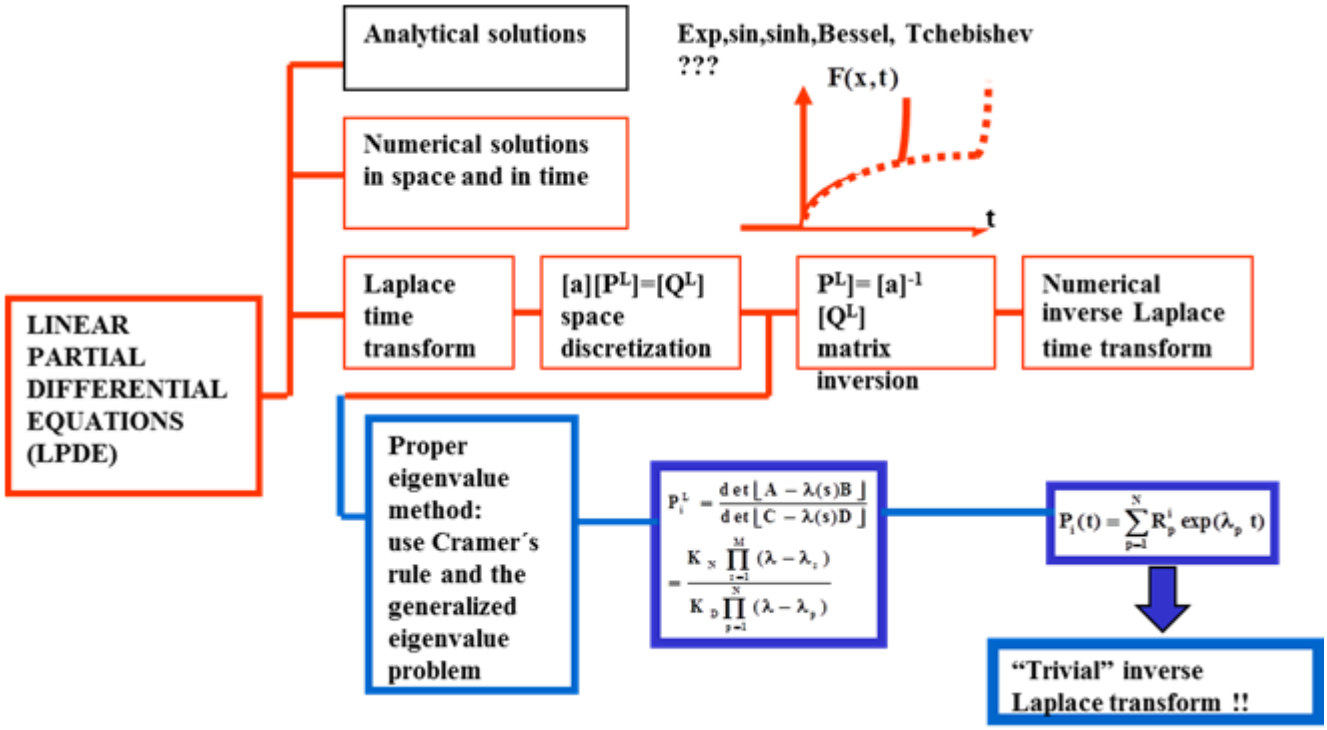

Figure 1: Possible solutions for Linear Partial Differential Equations

The third approach involves the use of Laplace time transforms leading to the following space discretized process:

$$[a][P^L] = [Q^L]$$
$$[P^L] = [a]^{-1}[Q^L]$$

Thus the time solution implies obtaining the inverse Laplace transform for the Laplace of all the P variables. Unfortunaly computer programs for the calculation of inverse Lapalace transforms are efficient when the systems have a limited number of poles and zeros.

Our approach is based on the application of Cramer rules for the solution of each Laplace transform for the P vaiables. Thus:

$$[P^L] = \frac{\det|num|}{\det|a|}$$

$$P_i^L = \frac{\det|A-\lambda(s)B|}{\det|C-\lambda(s)D|} = \frac{K_n \sum_1^M (\lambda - \lambda_z)}{K_d \sum_1^N (\lambda - \lambda_p)}$$

Where the poles and zeros of the system are determined by solving the following two generalized eigenvalue problems [Moler and Stewart 1973] and [Kaufman 1975]:

$$\det\ \left[\,[b]-\lambda[c]\,\right]=0 \ \Rightarrow\ \Rightarrow \qquad \lambda_{num}$$
$$\det\ \left[\,[a]-\lambda[I]\,\right]=0 \ \Rightarrow\ \Rightarrow \qquad \lambda_{den}$$

The ratio of the large determinants can then be evaluated term by term, once the eigenvalues are found. The inverse Laplace transform of any variable at different places in space is then determined by a "trivial" residue expression:

$$P_i(t) = \sum_{i=1}^{i=N} R_p^i \exp(\lambda_p t)$$

This expression allows the determination of the variable $P_i(t)$ at any time.

2. PROPER EIGENVALUE SOLUTION FOR A SIMPLE HEAT TRANSFER PROBLEM

We will show the PEM solution for the very simple one-dimensional problem shown in Figure 2. We consider and infinitely long metal wire (along the z axis) of radius r_a, with thermal conductivity K_m and themal diffusivity κ_m. The wire is coated with an insulation layer of external radius r_b, with thermal conductivity K and thermal diffusivity κ. A step of electrical power with volumetric density P(t) is applied uniformly in the metal.

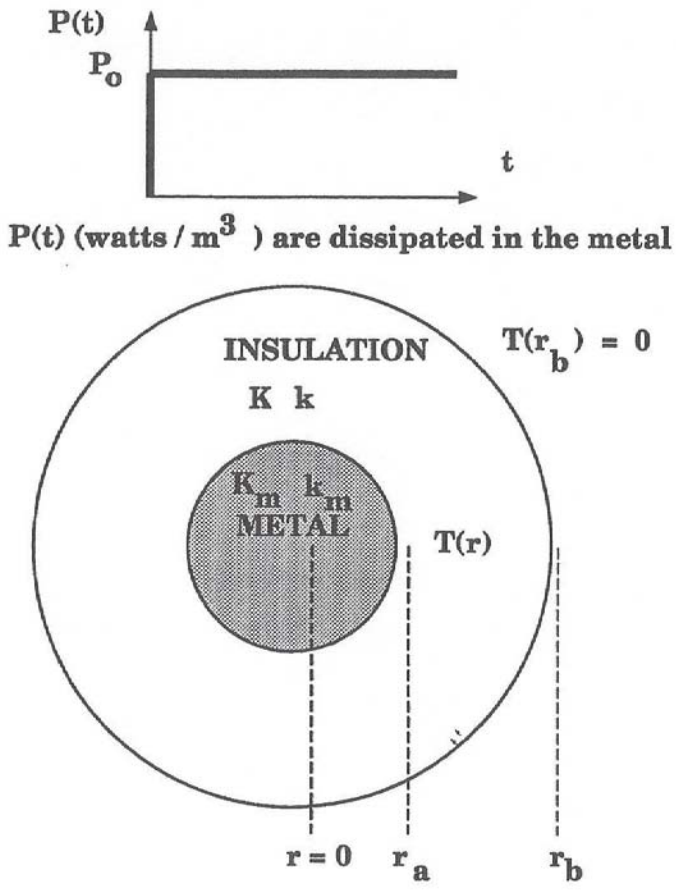

Figure 2: Geometry for a metal cable covered by insulation

The system is initially at room temperature and the outer surface is kept at 0 temperature. The heat trasnfer equations to be solved are:

$$\vec{\nabla}^2(T(r,t)) = \frac{1}{\kappa}\frac{\partial T(r,t)}{\partial t} \qquad \textbf{insulator}$$

$$\vec{\nabla}^2(T_m(r,t)) = \frac{1}{\kappa_m}\frac{\partial T_m(r,t)}{\partial t} - \frac{P(t)}{K_m} \qquad \textbf{metal}$$

Whose Lapace time transforms are:

$$\frac{d^2T^L(r)}{dr^2} + \frac{1}{r}\frac{dT^L(r)}{dr} = \frac{s}{\kappa}T^L(r) \qquad \textbf{insulator}$$

$$\frac{d^2T_M^L(r)}{dr^2} + \frac{1}{r}\frac{dT_M^L(r)}{dr} = \frac{s}{\kappa_M}T^L(r) - \frac{P_0}{sK_m} \qquad \textbf{metal}$$

The baundary conditions are:

$$T_M(0) = \text{finite} \qquad T_M(r_a) = T(r_a) \qquad T(r_b) = 0$$

$$-K_m\left[\frac{dT_M^L(r)}{dr}\right]_{ra} = -K\left[\frac{dT^L(r)}{dr}\right]_{rb}$$

The equivalent circuit for the system will then be the circuit shown on Figure 3.

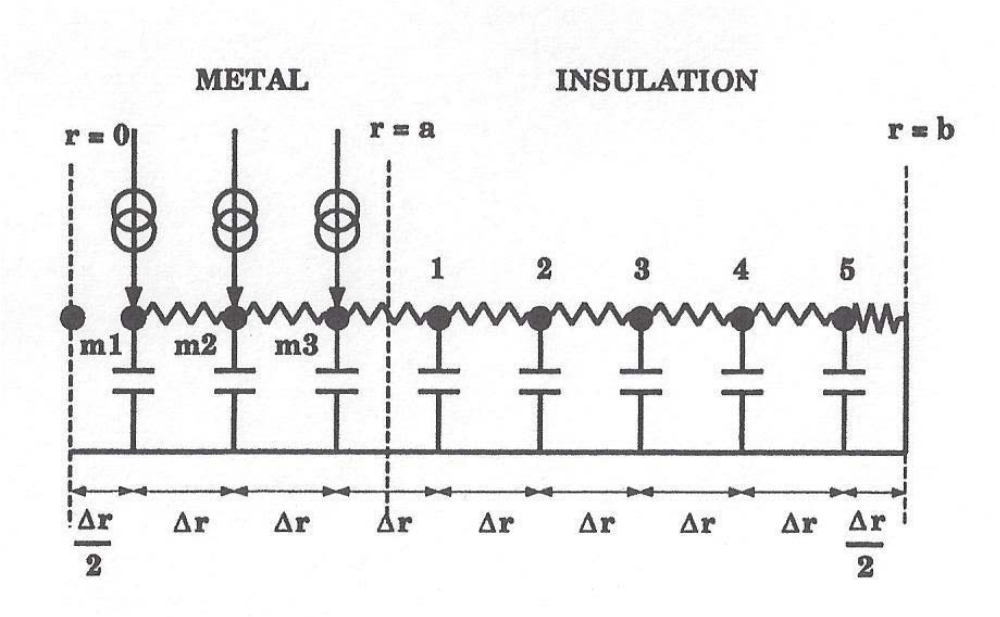

Figure 3: Equivalent circuit for 8 nodes

If the thermal conductivity of the metal is sufficiently large, for values of time larger than t_{min}, where:

$$t_{min} \equiv \frac{a^2}{\kappa_M}$$

Then we can consider the insulation region alone with the following equivalent circuit:

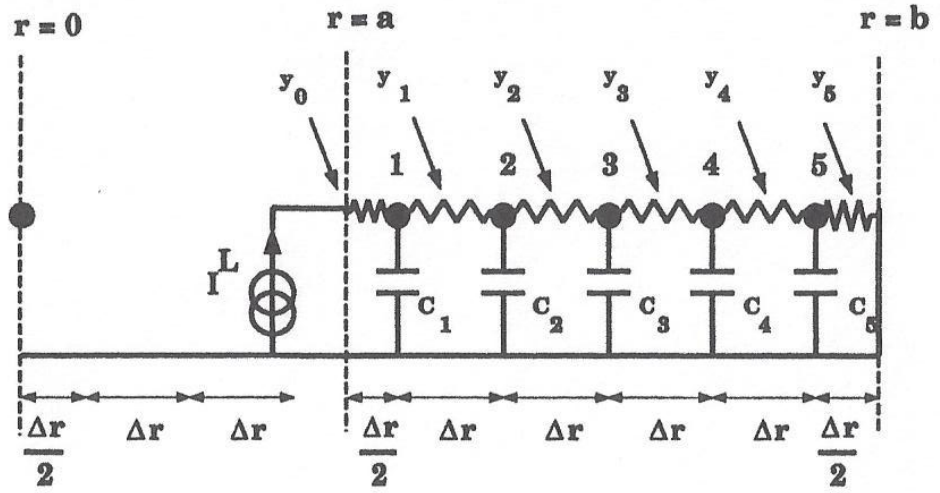

Figure 4: Reduced equivalent circuit with 5 nodes

Where for a length Δz:

$$I^L = (\frac{P_0}{s})\ \pi\ r_a^2(\Delta z)$$

The analystical solutions in terms of Bessel fuctions are:

$$T_M^L(r) = (\frac{\kappa_M P_0}{K_M s^2}) + A\ J_0(p_M r) \qquad p_M = \sqrt{\frac{-s}{\kappa_M}}$$

$$T^L(r) = B\ J_0(pr) + C Y_0(pr) \qquad p = \sqrt{\frac{-s}{\kappa}}$$

Applying boundary conditions at the metal insulator interface relates C and B, and the remaining two constants are given by:

$$\begin{bmatrix} -J_0(p_M a) & \left\{J_0(pa) - \frac{J_0(pb)}{Y_0(pb)} Y_0(pa)\right\} \\ K_M p_M J_1(p_M a) & -Kp\left\{J_1(pa) - \frac{J_0(pb)}{Y_0(pb)} Y_1(pa)\right\} \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} \frac{\kappa_M P_0}{K_M s^2} \\ 0 \end{bmatrix}$$

Thus we find the temperature at r=a, as:

$$T^{L}(a)=\left[\frac{aP_0}{2K}\right]\left\{\frac{J_0(pa)Y_0(pb)-J_0(pb)Y_0(pa)}{sp\left[J_1(pa)Y_0(pb)-J_0(pb)Y_1(pa)\right]}\right\}$$

And in terms of the residues and the poles of the system:

$$T^{L}(a)=\left[\frac{a^2P_0}{2K}\right]\left\{\ln(\frac{b}{a})\sum_{i=1}^{\infty}\frac{J_0(\eta_i)J_1(\eta_i a/b)}{\eta\left\{J^2{}_0(\eta_i)-J\left(\eta_i a/b\right)\right\}}R\left(\eta_i a/b\right)\exp(-\eta_i^2\kappa t\right\}$$

$$R\left(\eta_i a/b\right)=J_0(\eta_i a/b)Y_0(\eta_i)-J_0(\eta_i)Y_0(\eta_i a/b)$$

$$\eta_i \quad \text{are the roots of} \quad \left\{J_1(\eta_i a/b)Y_0(\eta_i)-J_0(\eta_i)Y_0(\eta_i a/b)\right\}=0$$

Figure 5 shows the comparison between the steady state analytical solution vs. the circuital solution. 100 space elements with a/b=0.5.

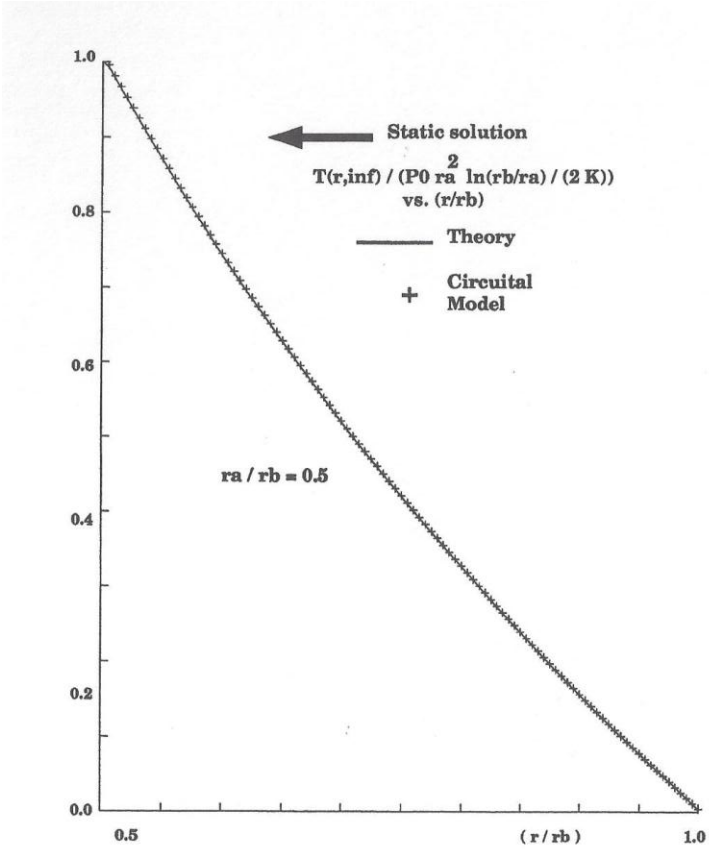

Figure 5: Steady state theoretical response vs. equivalent circuit response

We will now solve the problem with the proper eigenvalue procedure. For the circuit of Figure 4 we have:

$$(\Delta r) = \frac{r_b - r_a}{1 + i_{max}}$$

$$r_i = r_a + 0.5(\Delta r) + (i-1)(\Delta r)$$

$$C_i = 2K(\Delta z)\pi(\Delta r)r_i$$

$$Y_{i=1,4} = \frac{2K(\Delta z)\pi}{\ln\left[\frac{r_i + \Delta r}{r_i}\right]} \qquad Y_0 = \frac{2K(\Delta z)\pi}{\ln\left[\frac{r_i + 0.5\Delta r}{r_i}\right]} = Y_5$$

The matrix which describes the system is:

$$\begin{bmatrix} a_1 + sC_1 & a_{12} & 0 & 0 & 0 \\ a_{12} & a_2 + sC_1 & a_{23} & 0 & 0 \\ 0 & a_{23} & a_3 + sC_1 & a_{34} & 0 \\ 0 & 0 & a_{34} & a_4 + sC_1 & a_{45} \\ 0 & 0 & 0 & a_{45} & a_5 + sC_1 \end{bmatrix} \begin{bmatrix} T_1^L \\ T_2^L \\ T_3^L \\ T_4^L \\ T_5^L \end{bmatrix} = \begin{bmatrix} I^L \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$a_1 = y_1 \qquad a_{12} = -y_1 \qquad a_2 = y_1 + y_2 \qquad a_{23} = -y_2 \qquad \text{etc.}$$

We want to solve for $T_1{}^L$ by Cramer determinantal rule:

$$T_1^L = I^L \frac{\det\begin{bmatrix} a_2 + sC_1 & a_{23} & 0 & 0 \\ a_{23} & a_3 + sC_1 & a_{34} & 0 \\ 0 & a_{34} & a_4 + sC_1 & a_{45} \\ 0 & 0 & a_{45} & a_5 + sC_1 \end{bmatrix}}{\det\begin{bmatrix} a_1 + sC_1 & a_{12} & 0 & 0 & 0 \\ a_{12} & a_2 + sC_1 & a_{23} & 0 & 0 \\ 0 & a_{23} & a_3 + sC_1 & a_{34} & 0 \\ 0 & 0 & a_{34} & a_4 + sC_1 & a_{45} \\ 0 & 0 & 0 & a_{45} & a_5 + sC_1 \end{bmatrix}} \equiv I^L(s)\frac{|a_n - \lambda b_n|}{|a_d - \lambda b_d|}$$

Where we defined λ= -s, so that we can recognize the ratio of two generalized eigenvalue problems. The programs for the solution for these problems are available [Moler and Stewart 1973] and [Kaufman 1975]. The Laplace transform of T_1 is given by:

$$T_1^L = I^L(s)\frac{k_n \prod_{i=1to4}(\lambda - \lambda_i)}{k_d \prod_{j=1to5}(\lambda - \lambda_j)} = I^L(s)\frac{k_n \prod_{i=1to4}(s - z_i)}{k_d \prod_{j=1to5}(s - p_j)} = I^L(s)\frac{1}{c_1}\frac{\prod_{i=1to4}(s - z_i)}{\prod_{j=1to5}(s - p_j)}$$

$$T_1(t) = -\frac{P_0 \pi r_a^2}{c_1}\sum_{k=1}^{6} R_k \exp(p_k \kappa t)$$

$$R_k = \left[\frac{(s - s_k)\prod_{i=1to4}(s - z_i)}{s\prod_{j=1to5}(s - p_j)}\right]_{s=sk}$$

The curve shown in Figure 6, shows that the theoretical calculation with 400 roots for the Bessel functions required 77 seconds of computer time, while a standard finite difference solution (not discussed in detail in this paper)required 275 seconds, and the calculation based on the proper eigenvalue method required only 13 seconds.

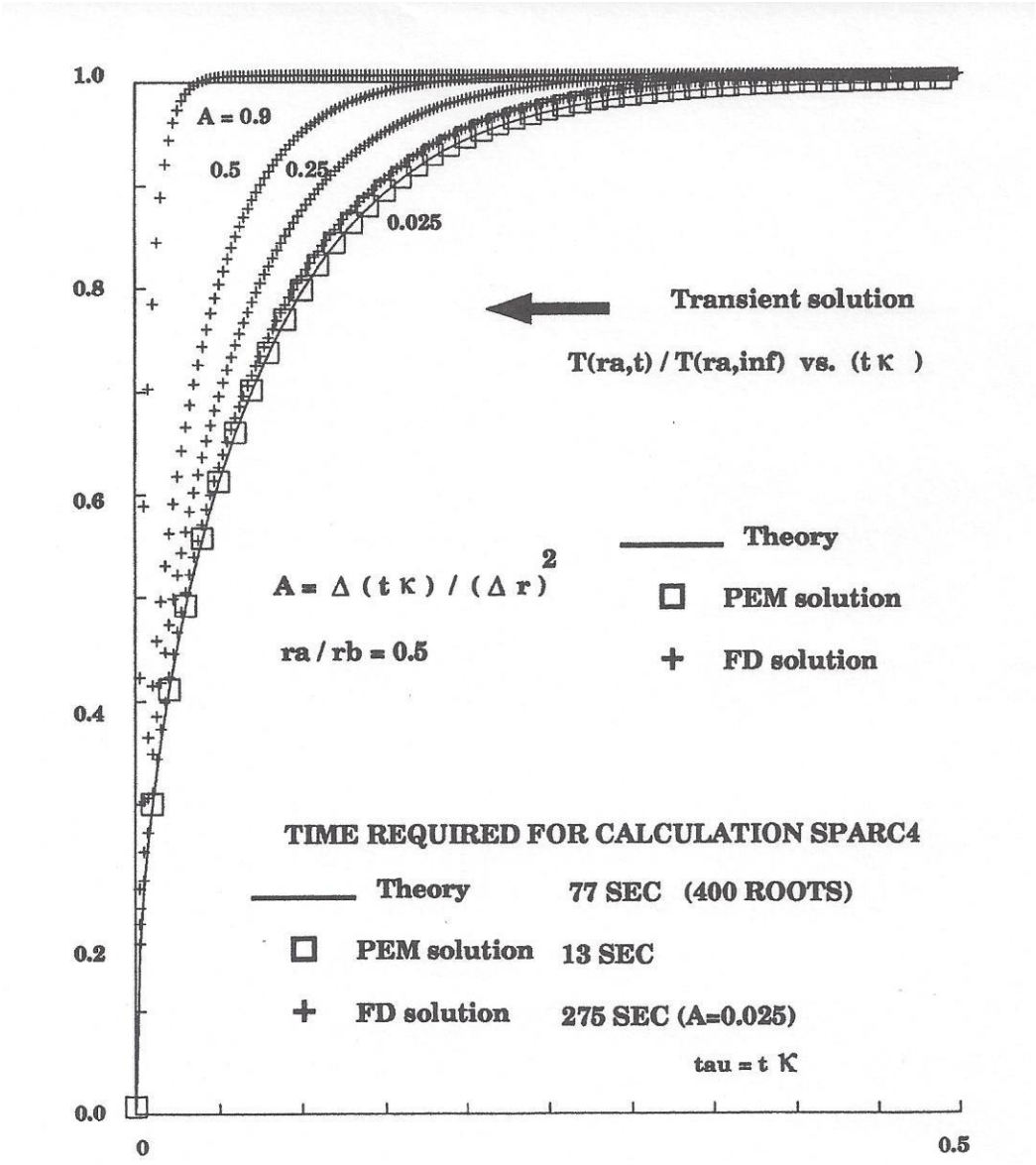

Figure 6: Comparison of the solution of the heat transfer problem by different methods: theory (continuous line), Finite difference solution (crosses), and proper eigenvalue method (squares).

3. RESULTS FOR ADDITIONAL COMPLEX SYSTEMS

We have shown the validity of the eigenvalue approach for a number of quite different fields:

a) heat transfer [Callarotti 1995][Callarotti 2012]
b) fluid flow in porous media [Callarotti 1995}
c) electromagnetic theory [Callarotti 1995] [Callarotti 1996] [Vera et al 1998]
d) electrochemistry [Yepez et al 1997] [Hernandez et al 1997]
e) with spectral methods [Callarotti 1998]
f) to engineering [Callarotti 2000]
g) to intelligent instrumentation [Callarotti 2004]
h) and to low velocity fluid mechanics [Callarotti 2005] [Callarotti 2011]

4. CONCLUSIONS

We feel that this method can be very helpful in engineering and science modeling of linear systems. The usual electrical engineering pole and zero analysis can be easily expanded to other disciplines, as an efficient common language. The next step in this area is the further possible reduction in required computer time by the formulation with parallel algorithms.

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